INVARIANTS OF REFLECTION GROUPS, ARRANGEMENTS, AND NORMALITY OF DECOMPOSITION CLASSES IN LIE ALGEBRAS

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Abstract. Suppose that W is a finite, unitary, reflection group acting on the complex vector space V and X is a subspace of V. Define N to be the setwise stabilizer of X in W, Z to be the pointwise stabilizer, and $C = N/Z$. Then restriction defines a homomorphism from the algebra of W-invariant polynomial functions on V to the algebra of C-invariant functions on X. In this note we consider the special case when W is a Coxeter group, V is the complexified reflection representation of W, and X is in the lattice of the arrangement of W, and give a simple, combinatorial characterization of when the restriction mapping is surjective in terms of the exponents of W and C. As an application of our result, in the case when W is the Weyl group of a semisimple, complex, Lie algebra, we complete a calculation begun by Richardson in 1987 and obtain a simple combinatorial characterization of regular decomposition classes whose closure is a normal variety.

1. Introduction

Suppose that W is a finite, complex reflection group acting on the complex vector space $V = \mathbb{C}^l$ and X is a subspace of V. Define $N_X = \{ w \in W \mid w(X) = X \}$, the setwise stabilizer of X in W and $Z_X = \{ w \in W \mid w(x) = x \, \forall x \in X \}$, the pointwise stabilizer of X in V. Then Z_X is a normal subgroup of N_X and we set $C_X = N_X/Z_X$. It is easy to see that restriction defines a homomorphism from the algebra of W-invariant polynomial functions on V to the algebra of C_X-invariant functions on X, say $\rho\colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$. In this note we consider the special case when W is a Coxeter group, V is the complexified reflection representation of W, and X is in the lattice of the arrangement of W. Our main result is a simple combinatorial characterization in terms of the exponents of W and C_X of when the map ρ is surjective.

As an application, our main result combined with a theorem of Richardson [Ric87] leads immediately to a complete, and easily computable, classification of the regular decomposition classes in a complex, semisimple Lie algebra whose closure is a normal variety.

2. Statement of the main results

By a *hyperplane arrangement* we mean a pair $(V, \mathcal{A})$, where $\mathcal{A}$ is a finite set of hyperplanes in V. The arrangement of a subgroup $C \subseteq \mathrm{GL}(V)$ consists of the reflecting hyperplanes of the elements in C that act on V as reflections. We denote the arrangement of C in V by

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$\mathcal{A}(V, C)$. Define C^{ref} to be the subgroup generated by the reflections in C. Then obviously $\mathcal{A}(V, C) = \mathcal{A}(V, C^{\mathrm{ref}})$.

For general information about arrangements and reflection groups we refer the reader to [OT92] and [Bou68].

Suppose from now on that W is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. Unless otherwise specified, we allow the case when the generators of W are "pseudo-reflections," that is, elements in $\mathrm{GL}(V)$ with finite order whose 1-eigenspace is a hyperplane in V. For a subspace X of V we have two natural hyperplane arrangements in X:

- The restricted arrangement $\mathcal{A}(V, W)^X$ consisting of intersections $H \cap X$ for H in $\mathcal{A}(V, W)$ with $X \not\subseteq H$.
- The reflection arrangement $\mathcal{A}(X, C_X) = \mathcal{A}(X, C_X^{\mathrm{ref}})$ consisting of the reflecting hyperplanes of elements in C_X that act on X as reflections.

For a free hyperplane arrangement $\mathcal{A}$ we denote the multiset of exponents of $\mathcal{A}$ by $\exp(\mathcal{A})$. Terao [Ter80] has shown that reflection arrangements are free and that $\exp(\mathcal{A}(V, W)) = \operatorname{coexp}(W)$, where $\operatorname{coexp}(W)$ denotes the multiset of coexponents of W.

The lattice of a hyperplane arrangement is the set of subspaces of V of the form $H_1 \cap \cdots \cap H_n$ where $\{H_1, \ldots, H_n\}$ is a subset of $\mathcal{A}$. It is known that $\mathcal{A}(V, W)^X$ is free when W is a Coxeter group and X is a subspace in the lattice of $\mathcal{A}(V, W)$ (see [OT93], [Dou99]). Thus, in this case, we have that (1) $\exp(\mathcal{A}(X, C_X))$, $\exp\left(\mathcal{A}(V, W)^X\right)$, and $\exp(\mathcal{A}(V, W))$ are all defined; (2) $\exp(\mathcal{A}(X, C_X)) = \exp(C_X^{\mathrm{ref}})$; and (3) $\exp(\mathcal{A}(V, W)) = \exp(W)$.

We can now state our main result.

Theorem 2.1. *Suppose W is a finite Coxeter group, V affords the reflection representation of W, and X is in the lattice of the arrangement $\mathcal{A}(V, W)$. Then the restriction mapping $\rho \colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$ is surjective if and only if*

$$\exp(\mathcal{A}(X, C_X)) = \exp\left(\mathcal{A}(V, W)^X\right) \subseteq \exp(\mathcal{A}(V, W)).$$

To simplify the notation, in the rest of this paper we denote the arrangements $\mathcal{A}(X, C_X)$, $\mathcal{A}(V, W)^X$, and $\mathcal{A}(V, W)$ by $\mathcal{A}(C_X)$, $\mathcal{A}^X$, and $\mathcal{A}$ respectively.

In the next section, using a modification of an argument of Denef and Loeser [DL95], we show in Proposition 3.1 that if W is any complex reflection group, X is in the lattice of $\mathcal{A}$, $C_X = C_X^{\mathrm{ref}}$, and ρ is surjective, then $\mathcal{A}(C_X) = \mathcal{A}^X$ and $\exp(C_X) \subseteq \exp(W)$. It then follows that in this case $\mathcal{A}^X$ is a free arrangement, $\exp(A(C_X)) = \exp(\mathcal{A}^X)$, and $\exp(C_X) \subseteq \exp(W)$. In particular, the forward implication in the theorem holds whenever C_X acts on X as a reflection group.

In §4 we complete the proof of Theorem 2.1 by (1) showing in Proposition 4.1 that if W is a Coxeter group and C_X does not act on X as a reflection group, then ρ is not surjective and (2) computing all cases in which $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$ for a Coxeter group W and showing that ρ is surjective in these cases.

Notice that the conditions $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$ are not that easy to satisfy. In case W is a Coxeter group of type A_{r-1}, up to the action of W, the subspaces X in the lattice

of $\mathcal{A}$ are parametrized by partitions of r. The conditions $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$ hold if and only if the corresponding partition of r has equal parts. For W a Coxeter group of type E_8, up to the action of W, there are forty-one possibilities for X, eight of which have the property that $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$. All cases in which $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$ when W is a finite, irreducible Coxeter group are listed in Tables 1 and 2 in §4.

In the rest of this section we explain how our main result leads to a characterization of regular decomposition classes in a complex, semisimple Lie algebra whose closure is a normal variety. The classification of these decomposition classes was completed, case-by-case, for classical Weyl groups by Richardson in 1987 [Ric87] and extended by Broer in 1998 [Bro98], again using case-by-case arguments, to exceptional Weyl groups.

Suppose that $\mathfrak{g}$ is a semisimple, complex Lie algebra and G is the adjoint group of $\mathfrak{g}$. Motivated by a question of De Concini and Procesi about the normality of the closure of the G-saturation of a Cartan subspace for an involution of $\mathfrak{g}$, Richardson proved the following.

Theorem 2.2 ([Ric87, Theorem B])**.** *Suppose that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, W is the Weyl group of $(\mathfrak{g}, \mathfrak{t})$, and X is a subspace of $\mathfrak{t}$ with the property that C_X acts on X as a reflection group. Let Y denote the closure of the set of elements in $\mathfrak{g}$ whose semisimple part is in $\mathrm{Ad}(G)X$. Then Y is a normal, Cohen-Macaulay variety if and only if $\rho\colon \mathbb{C}[\mathfrak{t}]^W \to \mathbb{C}[X]^{C_X}$ is surjective.*

When $V = \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, a subspace X of $\mathfrak{t}$ is in the lattice of $\mathcal{A}(\mathfrak{t}, W)$ if and only if there is a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ with $\mathfrak{t} \subseteq \mathfrak{l}$ so that $X = \mathfrak{z}$ is the center of $\mathfrak{l}$.

Now let $\mathfrak{g}_{\mathrm{reg}}$ denote the set of regular elements in $\mathfrak{g}$. Then $\mathfrak{g}_{\mathrm{reg}}$ is the disjoint union of decomposition classes of $\mathfrak{g}$ (see [Bor81, §3]). A decomposition class contained in $\mathfrak{g}_{\mathrm{reg}}$ is a *regular decomposition class.* Suppose that $\mathfrak{l}$ and $\mathfrak{z}$ are as in the last paragraph, $\mathfrak{z}_0$ is the subspace of elements in $\mathfrak{z}$ whose centralizer in $\mathfrak{g}$ is $\mathfrak{l}$, and $\mathcal{O}$ is the regular, nilpotent, adjoint orbit in $\mathfrak{l}$. Then $\mathrm{Ad}(G)(\mathfrak{z}_0 + \mathcal{O})$ is a regular decomposition class. Moreover, every regular decomposition class is of this form for some $\mathfrak{l}$ [Bor81, §3]. Therefore, combining Theorems 2.1 and 2.2, we obtain the following characterization of regular decomposition classes in $\mathfrak{g}$ that have normal closure.

Theorem 2.3. *With the notation above, suppose that $D = \mathrm{Ad}(G)(\mathfrak{z}_0 + \mathcal{O})$ is a regular decomposition class in $\mathfrak{g}$. Then $\overline{D}$ is a normal variety if and only if*

$$\exp(\mathcal{A}(\mathfrak{z}, C_{\mathfrak{z}})) = \exp(\mathcal{A}(\mathfrak{t}, W)^{\mathfrak{z}}) \subseteq \exp(\mathcal{A}(\mathfrak{t}, W)).$$

Using case-by-case arguments Richardson [Ric87] determined all cases in which $\rho\colon \mathbb{C}[\mathfrak{t}]^W \to \mathbb{C}[\mathfrak{z}]^{C_{\mathfrak{z}}}$ is surjective when W is a Weyl group of classical type. Broer [Bro98] computed almost all of the additional cases for exceptional Weyl groups. The statement of [Bro98, Theorem 3.1 (e7)] is missing one case: If $\mathfrak{g}$ is of type E_7 and $\mathfrak{l}$ is of type $(A_1^3)'$ (with simple roots α_2, α_5, α_7, where the labeling is as in [Bou68]), then the restriction map ρ is surjective.

3. A preliminary result

In this section we prove the following result.

Proposition 3.1. *Suppose $W \subseteq \mathrm{GL}(V)$ is a complex reflection group, X is in the lattice of $\mathcal{A}$, C_X acts on X as a reflection group, and the restriction mapping $\rho\colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$ is surjective. Then $\exp(C_X) \subseteq \exp(W)$ and $\mathcal{A}(C_X) = \mathcal{A}^X$. Thus, $\mathcal{A}^X$ is a free arrangement and if W is a Coxeter group, then $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$.*

The proof shows that if X is any subspace of V, C_X acts on X as a reflection group, and ρ is surjective, then $\exp(C_X) \subseteq \exp(W)$ and $\mathcal{A}(C_X) \subseteq \mathcal{A}^X$. The assumption that X is in the lattice of $\mathcal{A}$ is only used to conclude that $\mathcal{A}^X \subseteq \mathcal{A}(C_X)$.

By assumption, the restriction mapping $\rho\colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$ is a degree-preserving, surjective homomorphism of graded polynomial algebras and so by a result of Richardson [Ric87, §4], we may choose algebraically independent, homogeneous polynomials $f_1, \ldots, f_r$ in $\mathbb{C}[V]^W$ so that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_r]$ and $\mathbb{C}[X]^{C_X} = \mathbb{C}[\rho(f_1), \ldots, \rho(f_l)]$. Since $\exp(C_X) = \{\deg f_1 - 1, \ldots, \deg f_l - 1\}$ and $\exp(W) = \{\deg f_1 - 1, \ldots, \deg f_r - 1\}$, we have $\exp(C_X) \subseteq \exp(W)$.

We next show that $\mathcal{A}(C_X) \subseteq \mathcal{A}^X$. Suppose K is in $\mathcal{A}(C_X)$. By assumption there is a w in N_X so that $\mathrm{Fix}(w) \cap X = K$. It is shown in [OT92, Theorem 6.27] that $\mathrm{Fix}(w)$ is in the lattice of $\mathcal{A}$, say $\mathrm{Fix}(w) = H_1 \cap \cdots \cap H_n$, where $H_1, \ldots, H_n$ are in $\mathcal{A}$. Then $K = H_1 \cap \cdots \cap H_n \cap X$. Since $\dim K = \dim X - 1$, it follows that $K = H_i \cap X$ for some i and so K is in $\mathcal{A}^X$.

It remains to show that $\mathcal{A}^X \subseteq \mathcal{A}(C_X)$. We use a variant of an argument given by Denef and Loeser [DL95] (see also [LS99]).

Suppose that homogeneous polynomial invariants $\{f_1, \ldots, f_r\}$ have been chosen as above. Let J denote the $r \times r$ matrix whose (i,j) entry is $\frac{\partial f_i}{\partial x_j}$ and let J_1 denote the $l \times l$ submatrix of J consisting of the first l rows and columns. Then J and J_1 are matrices of functions on V. For v in V, let $J(v)$ and $J_1(v)$ be the matrices obtained from J and J_1 respectively by evaluating each entry at v.

Then $\det J_1$ is in $\mathbb{C}[V]$ and by a result of Steinberg (see [OT92, §6.2]) the zero set of $\rho(\det J_1) = \det \rho(J_1)$ in X is precisely $\bigcup_{K \in \mathcal{A}(C_X)} K$. Thus, to show that $\mathcal{A}^X \subseteq \mathcal{A}(C_X)$ it is enough to show that if K is in $\mathcal{A}^X$, then $\rho(\det J_1)$ vanishes on K.

Denef and Loeser have shown that if w is in W, v_1 and v_2 are eigenvectors for w with eigenvalues λ_1 and λ_2 respectively, and f in $\mathbb{C}[V]^W$ is homogeneous with degree d, then $\lambda_2 D_{v_2}(f)(v_1) = \lambda_1^{1-d} D_{v_2}(f)(v_1)$, where $D_v(f)$ denotes the directional derivative of f in the direction of v. This proves the following lemma.

Lemma 3.2. *Suppose w is in W, x is in $\mathrm{Fix}(w)$ and v in V is an eigenvector of w with eigenvalue $\lambda \neq 1$. Then $D_v(f)(x) = 0$ for every f in $\mathbb{C}[V]^W$.*

Suppose H is in $\mathcal{A}$, s is a reflection in W that fixes H, and v is orthogonal to H with respect to some W-invariant inner product on V. Since H is the full 1-eigenspace of s in V, Lemma 3.2 shows that

$$D_v(f) \quad \text{vanishes on } H \text{ for every } f \text{ in } \mathbb{C}[V]^W. \tag{3.3}$$

By [OT92, Theorem 6.27], we may find w in W with $\mathrm{Fix}(w) = X$. Choose a basis $\{b_1, \dots, b_r\}$ of V consisting of eigenvectors for w so that $\{b_1, \dots, b_l\}$ is a basis of X. Let $\{x_1, \dots, x_r\}$ denote the dual basis of V^*. Since X is the full 1-eigenspace of w in V, Lemma 3.2 shows that

$$\text{for } j > l,\ D_{b_j}(f) = \frac{\partial f}{\partial x_j} \text{ vanishes on } X \text{ for every } f \text{ in } \mathbb{C}[V]^W. \tag{3.4}$$

Now suppose K is in $\mathcal{A}^X$. Say $K = H \cap X$, where H is in $\mathcal{A}$ with $X \not\subseteq H$. Choose v in V orthogonal to H with respect to a W-invariant inner product. Say $v = \sum_{i=1}^r \xi_i b_i$. Define $[v]$ to be the column vector whose i^{th} entry is ξ_i for $1 \leq i \leq r$ and $[v_1]$ to be the column vector whose i^{th} entry is ξ_i for $1 \leq i \leq l$. It follows from (3.3) that $J(h) \cdot [v] = 0$ for every h in H. Therefore, it follows from (3.4) that $J_1(k) \cdot [v_1] = 0$ for every k in K. Since $X \not\subseteq H$, we have $[v_1] \neq 0$ and so it must be the case that for k in K, the matrix $J_1(k)$ is not invertible. Therefore, $\det J_1$ vanishes on K and so $\rho(\det J_1)$ vanishes on K. Thus, K is in $\mathcal{A}(C_X)$. This completes the proof of Proposition 3.1.

4. Completion of the proof of Theorem 2.1

In this section we complete the proof of Theorem 2.1 and show that if W is a Coxeter group, V affords the reflection representation of W, and X is in the lattice of $\mathcal{A}$, then $\rho\colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$ is surjective if and only if $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$.

In the arguments below, "degree" means with respect to the natural grading on $\mathbb{C}[V]$. For an integer d, let $\mathbb{C}[V]_d$ denote the subspace of elements of degree d. For a subalgebra R of $\mathbb{C}[V]$ we set $R_d = R \cap \mathbb{C}[V]_d$. After choosing an appropriate basis of V we may consider $\mathbb{C}[X]$, $\mathbb{C}[X]^{C_X}$, and $\mathbb{C}[X]^{C_X^{\text{ref}}}$ as subalgebras of $\mathbb{C}[V]$.

Also, we use the conventions that in type A, A_{-1} and A_0 are to be interpreted as the trivial group; in type B, B_0 is to be interpreted as the trivial group and B_1 is to be interpreted as a component of type A_1 supported on a short root; and in type D, D_1 is to be interpreted as the trivial group and D_2 is to be interpreted as a component of type $A_1 \times A_1$ supported on the two distinguished end nodes in the Coxeter graph.

It is easy to see that if $W = W_1 \times W_2$ is reducible, then Theorem 2.1 holds for W if and only if it holds for W_1 and W_2. Thus, we may assume that W is an irreducible Coxeter group.

Fix a generating set S in W so that (W, S) is a Coxeter system. For a subset I of S define $X_I = \cap_{s \in I} \mathrm{Fix}(s)$ and $W_I = \langle I \rangle$, the subgroup of W generated by I. Orlik and Solomon [OS83] have shown that there is a w in W and a subset I of S so that $w(X) = X_I$, $wZ_Xw^{-1} = W_I$, and $wN_Xw^{-1} = N_W(W_I)$. Howlett [How80] has shown that W_I has a canonical complement, C_I, in $N_W(W_I)$.

We say that *C_I acts on X_I as a Coxeter group with full rank* if $C_I = C_I^{\text{ref}}$ and the Coxeter rank of C_I equals the dimension of X_I. For example, if W is of type E_6 and W_I is of type $A_1 \times A_2$, then $C_I = C_I^{\text{ref}}$ is of type A_2 and $\dim X_I = 3$, so C_I does not act on X_I as a Coxeter group with full rank. Another example is when W is of type $I_2(r)$ with r odd and I is a one element subset of S. In this case, C_I is the trivial group and X_I is one-dimensional.

Suppose now that the restriction mapping ρ is surjective. It follows from the next proposition that C_X acts on X as a Coxeter group with full rank. In particular, we may apply Proposition 3.1 and conclude that $\exp(\mathcal{A}(C_X)) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$. This proves the forward implication of Theorem 2.1.

Proposition 4.1. *Suppose W is a Coxeter group, X is in the lattice of $\mathcal{A}$, and C_X does not act on X as a Coxeter group with full rank. Then the restriction mapping $\rho\colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$ is not surjective.*

Proof. We may assume that W is irreducible and that $X = X_I$ for some subset I of S. Then $W_X = W_I$, $N_X = N_W(W_I)$, and $C_X = C_I$. To show that ρ is not surjective, in each case when C_I does not act on X_I as a Coxeter group with full rank, we find an integer d so that $\dim \mathbb{C}[V]^W_d < \dim \mathbb{C}[X_I]^{C_I}_d$. It then follows that $\mathbb{C}[X_I]^{C_I}_d$ is not contained in the image of ρ.

If $I = \emptyset$ or $I = S$, then C_I acts on X_I as a Coxeter group with full rank. Thus, we may assume that I is a non-empty, proper subset of S.

Howlett [How80] has computed C_I, C_I^{ref}, and the representation of C_I on X_I for all Coxeter groups with rank greater than two. When W has rank two, W is of type $I_2(r)$ for some r. It is easy to see that in this case C_I acts on X_I as a Coxeter group with full rank unless r is odd and $|I| = 1$. Then, as noted above, C_I is the trivial group acting on the one-dimensional vector space X_I.

The subgroup C_I^{ref} is always a normal subgroup of C_I and it turns out that if $C_I^{\mathrm{ref}} \neq C_I$, then C_I is the semidirect product of C_I^{ref} with an elementary abelian 2-group. Notice that if w is any element in C_I with order two, then w acts on X_I with eigenvalues ± 1, and so w fixes every even degree, homogeneous, polynomial function on X_I. Therefore,

$$\mathbb{C}[X_I]^{C_I}_{2n} = \mathbb{C}[X_I]^{C_I^{\mathrm{ref}}}_{2n}$$

for all n. Consequently, if either C_I^{ref} is reducible or C_I^{ref} is irreducible and the Coxeter rank of C_I^{ref} is strictly less than the dimension of X_I, then $\dim \mathbb{C}[X_I]^{C_I}_2 > 1 = \dim \mathbb{C}[V]^W_2$ and so ρ is not surjective.

It remains to consider the cases when $C_I \neq C_I^{\mathrm{ref}}$, C_I^{ref} is irreducible, and the Coxeter rank of C_I^{ref} equals $\dim X_I$.

If W is a dihedral group, then $C_I = C_I^{\mathrm{ref}}$ for all I.

If W is of classical type and $C_I \neq C_I^{\mathrm{ref}}$, then W is of type D_r and W_I has only components of type A. Suppose that this is the case. Then it follows from Howlett's computations [How80] that whenever $C_I \neq C_I^{\mathrm{ref}}$, either C_I^{ref} is reducible or the Coxeter rank of C_I^{ref} is strictly less than the dimension of X_I.

There are four cases when $C_I \neq C_I^{\mathrm{ref}}$, C_I^{ref} is irreducible, and the Coxeter rank of C_I^{ref} equals $\dim X_I$: either W is of type E_7 and W_I is of type A_2, or W is of type E_8 and W_I is of type A_2, $A_1 \times A_2$, or A_4.

Suppose W is of type E_7 and W_I is of type A_2, or that W is of type E_8 and W_I is of type $A_1 \times A_2$. We show that $\dim \mathbb{C}[V]^W_4 < \dim \mathbb{C}[X_I]^{C_I}_4$. Fix $f_2 \neq 0$ in $\mathbb{C}[V]^W_2$. Because the

two smallest exponents of W are $1, 5$ and $1, 7$, respectively, it follows that $\mathbb{C}[V]_4^W$ is one-dimensional with basis $\{f_2^2\}$. Since C_I^{ref} is of type A_5 in both cases, we have $\dim \mathbb{C}[X_I]_4^{C_I} = \dim \mathbb{C}[X_I]_4^{C_I^{\mathrm{ref}}} = 2$.

Finally, suppose W is of type E_8 and W_I is of type A_2 or A_4. We show that $\dim \mathbb{C}[V]_6^W < \dim \mathbb{C}[X_I]_6^{C_I}$. Fix $f_2 \neq 0$ in $\mathbb{C}[V]_2^W$. Since the two smallest exponents of W are 1 and 7, it follows that $\mathbb{C}[V]_6^W$ is one-dimensional with basis $\{f_2^3\}$. Because C_I^{ref} is of type E_6 when W_I is of type A_2 and that C_I^{ref} is of type A_4 when W_I is of type A_4, we have $\dim \mathbb{C}[X_I]_6^{C_I} = 2$ in the first case, and $\dim \mathbb{C}[X_I]_6^{C_I} = 3$ in the second. This completes the proof of the proposition. $\square$

To complete the proof of Theorem 2.1 we suppose that $\exp(C_X) = \exp(\mathcal{A}^X) \subseteq \exp(\mathcal{A})$ and show that $\rho\colon \mathbb{C}[V]^W \to \mathbb{C}[X]^{C_X}$ is surjective. Our argument is case-by-case, using the computation of $\exp(\mathcal{A}^X)$ by Orlik and Solomon [OS83], Howlett's results in [How80], and some computer-aided computations using GAP [S+97] for six cases when W is of exceptional type. For W of classical type, our argument is similar to that of Richardson [Ric87], but more streamlined, especially when W is of type D_r, because of our assumptions on $\mathcal{A}^X$.

As above, we may assume that W is irreducible and that $X = X_I$ for some proper, non-empty, subset I of S. Then $W_X = W_I$, $N_X = N_W(W_I)$, and $C_X = C_I$. Notice that it follows from the assumption $\exp(C_I) \subseteq \exp(\mathcal{A})$ that C_I^{ref} is irreducible.

Suppose first that W is classical of type A_r, B_r, or D_r with $r \geq 1$, $r \geq 2$, and $r \geq 4$, respectively. Say W_I has m_i components of type A_i and a component of type B_j or D_j, where $j \geq 0$. In type A we set $j = -1$. Set $k = j + \sum_i (i+1)m_i$. Then k is minimal so that W_I may be embedded in a Coxeter group of type A_k, B_k, or D_k. The group C_I^{ref} is given as follows:

- $\prod_i A_{m_i-1} \times A_{r-k-1}$ if W is of type A_r,
- $\prod_i B_{m_i} \times B_{r-k}$ if W is of type B_r,
- $\prod_i B_{m_i} \times B_{r-k}$ if W is of type D_r and $j \neq 0$, and
- $\prod_{i \text{ even}} D_{m_i} \times \prod_{i \text{ odd}} B_{m_i} \times D_{r-k}$ if W is of type D_r and $j = 0$.

The exponents of $\mathcal{A}^{X_I}$ have been computed by Orlik and Solomon in [OS83]. Set $l = \dim X_I$. Then $\exp(\mathcal{A}^{X_I})$ is given as follows:

- $\{1, 2, 3, \ldots, l\}$ if W is of type A_r,
- $\{1, 3, 5, \ldots, 2l-1\}$ if W is of type B_r,
- $\{1, 3, 5, \ldots, 2l-1\}$ if W is of type D_r and $j \neq 0$, and
- $\{1, 3, 5 \ldots, 2l-3, l-1+\sum_i m_i\}$ if W is of type D_r and $j = 0$.

Type A_r. Suppose W is of type A_r. If $r-k-1 > 0$, then since C_I is irreducible it must be that $m_i \leq 1$ for all i. Then $\exp(C_I) = \{1, 2, \ldots, r - \sum_i (i+1)\}$ and $\exp(\mathcal{A}^{X_I}) = \{1, 2, \ldots, r - \sum_i i\}$, and so $r - \sum_i (i+1) = r - \sum_i i$, which is absurd. Therefore, $r-k-1 \leq 0$. Thus, $r \leq k+1$ and W_I is of type A_d^m. In this case, $\exp(C_I) = \{1, 2, \ldots, m-1\}$, $\dim X_I = r - dm$, and

$\exp(\mathcal{A}^{X_I}) = \{1, 2, \ldots, r - dm\}$. Therefore, $m - 1 = r - dm$. We conclude that $\exp(C_I) = \exp(\mathcal{A}^{X_I}) \subseteq \exp(\mathcal{A})$ if and only if W_I is of type A_d^m, where r, d, and m are related by the equation $r + 1 = (d + 1)m$.

Now suppose that W_I is of type A_d^m with $r + 1 = (d + 1)m$. Then identifying W with the symmetric group S_{r+1} acting on $\mathbb{C}^{r+1}$, V with the subspace of $\mathbb{C}^{r+1}$ consisting of all vectors whose components sum to zero, W_I with the Young subgroup $S_{d+1}^m \subseteq S_{r+1}$, and taking the power sums as a set of fundamental polynomial invariants for S_{r+1}, it is straightforward to check that ρ is surjective.

Type B_r. Suppose that W is of type B_r with $r \geq 2$. Since C_I is irreducible, there is at most one value of i with $m_i > 0$. Suppose first that there is a value of i with $m_i > 0$. Say W_I has type $A_d^m \times B_j$. Then we must have $r - k = 0$ and so r, j, d, and m are related by $r = j + (d + 1)m$. In this case, C_I has type B_m and $\dim X_I = r - j - dm = m$. Thus $\exp(C_I) = \{\,1, 3, \ldots, 2m{-}1\} = \exp(\mathcal{A}^{X_I})$. On the other hand, if $m_i = 0$ for all i, then W_I is of type B_j, C_I is of type B_{r-j}, $\dim X_I = r{-}j$, and $\exp(C_I) = \{1, 3, \ldots, 2(r{-}j){-}1\} = \exp(\mathcal{A}^{X_I})$. We conclude that $\exp(C_I) = \exp(\mathcal{A}^{X_I}) \subseteq \exp(\mathcal{A})$ if and only if W_I is of type $A_d^m \times B_j$, where if $m > 0$, then r, d, j, and m satisfy $r = j + (d + 1)m$.

Now suppose that W_I is of type $A_d^m \times B_j$ with $r = j + (d+1)m$ if $m > 0$. We may consider W as signed permutation matrices acting on $\mathbb{C}^r$. Let $x_1, \ldots, x_r$ denote the coordinate functions on $\mathbb{C}^r$. Then $\mathbb{C}[V]^W = \mathbb{C}[x_1, \ldots, x_r]^W = \mathbb{C}[f_2, f_4, \ldots, f_{2r}]$, where f_{2p} is the p^{th} elementary symmetric function in $\{x_1^2, \ldots, x_r^2\}$. In case $m > 0$, we may choose coordinate functions $\{y_1, \ldots, y_m\}$ on X_I so that C_I acts as signed permutations on the coordinates and the restriction map $\mathbb{C}[V] \to \mathbb{C}[X_I]$ is given by mapping $x_{p(d+1)+q}$ to y_p for $0 \leq p \leq m - 1$ and $1 \leq q \leq d + 1$, and x_t to zero for $t > r - j = (d + 1)m$. It is then easily checked that $\rho\colon \mathbb{C}[x_1, \ldots, x_r]^W \to \mathbb{C}[y_1, \ldots, y_m]^{C_I}$ is surjective. In case $m = 0$ we may take C_I to act on the first $r - j$ components of $\mathbb{C}^r$ and so the restriction map $\mathbb{C}[V] \to \mathbb{C}[X_I]$ is given by evaluating $x_{r-j+1}, \ldots, x_r$ at zero. It is now easily checked that $\rho\colon \mathbb{C}[x_1, \ldots, x_r]^W \to \mathbb{C}[x_1, \ldots, x_{r-j}]^{C_I}$ is surjective.

Type D_r. Suppose that W is of type D_r with $r \geq 4$. In case $j \neq 0$ the argument for type B applies almost verbatim (B_j is replaced by D_j) and shows that $\exp(C_I) = \exp(\mathcal{A}^{X_I}) \subseteq \exp(\mathcal{A})$ if and only if W_I is of type $A_d^m \times D_j$, where if $m > 0$, then r, d, j, and m satisfy $r = j + (d + 1)m$. In the case when $j = 0$, the arrangement $\mathcal{A}^{X_I}$ is a Coxeter arrangement if and only if either $\sum_i m_i = 0$, in which case it is a Coxeter arrangement of type D_l, or $\sum_i m_i = l$, in which case it is a Coxeter arrangement of type B_l. Since $\sum_i m_i \neq 0$, we must have that $\sum_i m_i = l = r - \sum_i i m_i$ and $\mathcal{A}^{X_I}$ is of type B_l. Thus, C_I^{ref} must be of type B_l and so W_I must be of type A_d^m, where d is odd and $r = (d + 1)m$. We conclude that if $j = 0$, then $\exp(C_I) = \exp(\mathcal{A}^{X_I}) \subseteq \exp(\mathcal{A})$ if and only if W_I is of type A_d^m, where d is odd and $r = (d + 1)m$.

Now suppose that W_I is of type $A_d^m \times D_j$, where if $j, m > 0$, then $r = j + (d + 1)m$, and if $j = 0$, then d is odd and $r = (d + 1)m$. We may consider W as signed permutation matrices with determinant 1 acting on $\mathbb{C}^r$. Then $\mathbb{C}[V]^W = \mathbb{C}[x_1, \ldots, x_r]^W = \mathbb{C}[f_2, f_4, \ldots, f_{2r-2}, g_r]$ where f_{2p} is the p^{th} elementary symmetric function in $\{x_1^2, \ldots, x_r^2\}$ and $g_r = x_1 \cdots x_r$. The

argument showing that ρ is surjective when W is of type B applies word for word to show that ρ is surjective in this case as well.

In order to determine the remaining cases when $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$, we fix a root system Φ for W. Then $\Phi \subseteq V^*$ and the choices of S and I determine a positive system and a closed parabolic subsystem denoted by Φ^+ and Φ_I, respectively. For α in Φ, we have $\alpha|_{X_I} \neq 0$ if and only if $\alpha \notin \Phi_I$.

If W_I is a maximal parabolic subgroup of W and $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$, then C_I is of type A_1 and acts as -1 on the one-dimensional space X_I. By [Bou68, Ch. VI §1.1], $f_2 = \sum_{\alpha\in\Phi} \alpha^2$ is a non-zero polynomial in $\mathbb{C}[V]^W_2$. Fix β in $\Phi^+ \setminus \Phi_I$. Then $\{\beta|_{X_I}\}$ is a basis of X_I^*. If $g_2 = \beta|_{X_I}^2$, then $\mathbb{C}[X_I]^{C_I} = \mathbb{C}[g_2]$. Since $\alpha|_{X_I}$ is a non-zero multiple of $\beta|_{X_I}$ for α in $\Phi^+ \setminus \Phi_I$, it follows that $\rho(f_2)$ is a non-zero multiple of g_2 and so ρ is surjective.

Suppose that W is of type $I_2(r)$ and $|I| = 1$. We have observed above that if r is odd, then C_I is the trivial group, so $\exp(C_I) = \{0\}$ and $\exp(\mathcal{A}^I) = \{1\}$. On the other hand, if r is even, then $\exp(C_I) = \exp(\mathcal{A}^I) = \{1\}$ and $\exp(\mathcal{A}) = \{1, m-1\}$ and so $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$.

Our computations when W is of classical or dihedral type are summarized in Table 1.

W	W_I	
A_r	A_d^m	$r+1 = (d+1)m$
B_r	$A_d^m B_j$	$m > 0 \Rightarrow r = j + (d+1)m$
D_r	$A_d^m D_j$	$[j, m > 0 \Rightarrow r = j + (d+1)m]$ or $[j = 0 \Rightarrow m$ odd $\wedge\ r = (d+1)m]$
$I_2(r)$	A_1, $\widetilde{A}_1$	r even

TABLE 1. Pairs (W, W_I) with W classical or dihedral, $\emptyset \neq I \neq S$, and $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$.

Finally, suppose that W is of exceptional type. The pairs (W, W_I) for which $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$ are given in Table 2. The notation is as in [OS83].

We have seen above that if W_I is maximal and $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$, then ρ is surjective. For the remaining six cases, A_2^2 in E_6; $(A_1^3)'$, $A_1^3 \times A_2$, and A_5' in E_7; and A_2 and $\widetilde{A}_2$ in F_4, the type of C_I is given in Table 3.

For these six cases, the fact that ρ is surjective was checked directly by implementing the following argument using GAP [S+97] and the CHEVIE package [GHL+96].

(1) For s in S let α_s and ω_s denote the simple root in V^* and the fundamental dominant weight in V^* determined by s respectively. Then $\{\, \omega_s \mid s \notin I \,\}$ is a basis of X_I^* and $\{\, \omega_s \mid s \notin I \,\} \cup \{\, \alpha_s \mid s \in I \,\}$ is a basis of V^*. This basis can be computed from the basis consisting of simple roots using the Cartan matrix of W. The restriction mapping $\mathbb{C}[V] \to \mathbb{C}[X_I]$ is then given by evaluating α_s at zero for s in I.

W	W_I									
E_6	A_2^2	$A_1A_2^2$	A_5							
E_7	$(A_1^3)'$	$A_1^3A_2$	A_5'	$A_1A_2A_3$	A_2A_4	A_1A_5	A_6	A_1D_5	D_6	E_6
E_8	$A_1A_2A_4$	A_3A_4	A_1A_6	A_7	A_2D_5	D_7	A_1E_6	E_7		
F_4	A_2	$\widetilde{A}_2$	C_3	B_3	$A_1\widetilde{A}_2$	$\widetilde{A}_1A_2$				
G_2	A_1	$\widetilde{A}_1$								
H_3	A_1A_1	A_2	$I_2(5)$							
H_4	A_1A_2	A_3	$A_1I_2(5)$	H_3						

TABLE 2. Pairs (W, W_I) with W of exceptional type, $\emptyset \neq I \neq S$, and $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$.

W	E_6	E_7			F_4	
W_I	A_2^2	$(A_1^3)'$	$A_1^3A_2$	A_5'	A_2	$\widetilde{A}_2$
C_I	G_2	F_4	G_2	G_2	G_2	G_2

TABLE 3. Triples (W, W_I, C_I) with W of exceptional type, $\emptyset \neq I$, $|I| < r-1$, and $\exp(C_I) = \exp(\mathcal{A}^I) \subseteq \exp(\mathcal{A})$.

(2) Suppose that the exponents of W are $\{d_1-1, d_2-1, \ldots, d_r-1\}$ where $\{d_1-1, d_2-1, \ldots, d_l-1\}$ are the exponents of C_I. For $i = 1, 2, \ldots, l$, define $f_i = \sum_{\alpha\in\Phi^+} \alpha^{d_i}$. Even though $\{f_1, \ldots, f_l\}$ is not obviously algebraically independent, each f_i is a non-zero element in $\mathbb{C}[V]^W_{d_i}$.

(3) For $i = 1, 2, \ldots, l$, express each f_i as a polynomial in $\{\, \omega_s \mid s \notin I \,\} \cup \{\, \alpha_s \mid s \in I \,\}$. Then set $\alpha_s = 0$ for s in I to get a polynomial $\rho(f_i)$ in $\mathbb{C}[X_I]^{C_I}_{d_i}$.

(4) Compute the Jacobian determinant of $\{\rho(f_1), \rho(f_2), \ldots, \rho(f_l)\}$.

It turns out that in all cases, the Jacobian determinant above is non-zero and so it follows from [Spr74, Prop. 2.3] that $\mathbb{C}[X_I]^{C_I} = \mathbb{C}[\rho(f_1), \rho(f_2), \ldots, \rho(f_l)]$. Therefore, ρ is surjective. This completes the proof of Theorem 2.1.

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